KUWAIT UNIVERSITY

Department of Mathematics

Math 102 June 6 2010

Calculus II Final Exam Time: 2 hrs.

Use of calculators is not allowed in this exam. Please switch off your mobile phones.

1. (1+2+1 pts.) Let $f(x) = \sqrt{(\ln x)^2 + 3}$

   (a) Show that f is one-to-one on $[1, \infty)$.

   (b) Find $(f^{-1})'(2)$.

   (c) Show that f is not one-to-one on $(0, \infty)$.

2. (2+2 pts.) Prove that

   (a) $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$ for all x and y in $\mathbb{R}$.

   (b) $\log_{1/x}(x) + \log_x(1/x) = -2$ for all $x > 1$.

3. (4 pts.) Evaluate $\lim_{x\to\infty}(e^{2x} + 3x)^{5/x}$.

4. (4+4 pts.) Evaluate the following integrals

   (a) $\int e^{3x} \cos x \, dx$ (b) $\int x^2\sqrt{1-x^2}\, dx$

5. (4 pts.) Determine whether the following improper integral is convergent or divergent. If it converges, find its value.

$$\int_0^4 \frac{dx}{x^2 + x - 6}$$

6. (4 pts.) Find the area of the surface obtained by rotating the curve $x = \sqrt{8y}$ for $y \in [0, 1]$ about the y-axis.

7. (4 pts.) Find the length of the curve given by $r = \cos^2(\theta/2)$ for $\theta \in [0, 2\pi]$.

8. (4 pts.) Consider the parametric equations $x = t\cos t$, $y = t\sin t$ for $t \in [0, 2\pi]$. Find an equation of the tangent line at $t = 3\pi/2$.

9. (2+2 pts.)

   (a) Sketch the polar curves $r = 2 - 2\cos\theta$ and $r = 2$.

   (b) Set up (but do not evaluate) the integral(s) representing the area inside the curves $r = 2 - 2\cos\theta$ and $r = 2$.

Q.1. a) $f'(x) = \dfrac{\ln x}{x\sqrt{\ln^2 x + 3}} \Rightarrow f'(x) > 0,\ \forall x > 1$

$\Rightarrow f$ is increasing and one-to-one on $[1, \infty)$.

b) $(f^{-1})'(2) = \dfrac{1}{f'(x)}$, where $2 = f(x)$ and $x \geq 1$.

$2 = \sqrt{\ln^2 x + 3} \Rightarrow \ln^2 x = 1 \Rightarrow \ln x = \pm 1 \Rightarrow x = e$

$\Rightarrow (f^{-1})'(2) = \dfrac{e\sqrt{\ln^2 e + 3}}{\ln e} = 2e$.

c) $f(e) = \sqrt{(\ln e)^2 + 3} = 2 = f(\frac{1}{e})$.

Q.2. a) $\cosh x \cosh y + \sinh x \sinh y = \frac{1}{4}(e^x + e^{-x})(e^y + e^{-y}) + \frac{1}{4}(e^x - e^{-x})(e^y - e^{-y})$

$= \frac{1}{4}(e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y} + e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y})$

$= \frac{1}{2}(e^{x+y} + e^{-x-y}) = \cosh(x+y)$.

b) $\log_{1/x}(x) + \log_x(1/x) = \dfrac{\ln x}{\ln(1/x)} + \dfrac{\ln(1/x)}{\ln x} = \dfrac{\ln x}{-\ln x} + \dfrac{-\ln x}{\ln x} = -2$.

Q.3. $y = (e^{2x} + 3x)^{5/x} \Rightarrow \ln y = \dfrac{5\ln(e^{2x} + 3x)}{x}$

$x \to \infty \Rightarrow e^{2x} + 3x \to \infty + \infty = \infty \Rightarrow \ln(e^{2x} + 3x) \to \infty$

$\left(\frac{\infty}{\infty}\right) \overset{\text{Hospital Rule}}{\Rightarrow} \lim\limits_{x\to\infty} \ln y = 5\lim\limits_{x\to\infty} \dfrac{2e^{2x} + 3}{e^{2x} + 3x} \overset{\left(\frac{\infty}{\infty}\right)}{=} 5\lim\limits_{x\to\infty} \dfrac{4e^{2x}}{2e^{2x} + 3} = 10$

$\Rightarrow \lim\limits_{x\to\infty} y = e^{10}$.

Q.4. a) $I = \int e^{3x} \cos x \, dx \overset{\text{parts}}{=} e^{3x} \sin x - 3J$, where

$J = \int e^{3x} \sin x \, dx \overset{\text{parts}}{=} -e^{3x} \cos x + 3I$

$\Rightarrow I = e^{3x} \sin x - 3(-e^{3x}\cos x + 3I)$

$\Rightarrow I = \frac{1}{10} e^{3x} (\sin x + 3\cos x)$.

b) $\int x^2 \sqrt{1 - x^2}\, dx \overset{x = \sin\theta}{=} \int \sin^2\theta \cos^2\theta \, d\theta = \frac{1}{4}\int \sin^2(2\theta)\, d\theta$

$= \frac{1}{8}\int (1 - \cos 4\theta)\, d\theta = \frac{1}{8}(\theta - \frac{1}{4}\sin 4\theta) + C$

$= \frac{1}{8}(\theta - \sin\theta \cos\theta \cos 2\theta) + C$

$= \frac{1}{8}\left[\sin^{-1} x - x\sqrt{1 - x^2}\,(1 - 2x^2)\right] + C$.

(2)

Q.5. $\dfrac{1}{x^2+x-6} = \dfrac{1}{(x+3)(x-2)}$ has V.A. at $x=2$ in $[0,4]$.

$$\int \frac{1}{(x+3)(x-2)}\,dx = \frac{1}{5}\int\left(\frac{1}{x-2}-\frac{1}{x+3}\right)dx = \frac{1}{5}\ln\left|\frac{x-2}{x+3}\right| + C$$

$$\Rightarrow \int_0^2 \frac{1}{x^2+x-6}\,dx = \lim_{t\to 2^-}\int_0^t \frac{dx}{(x+3)(x-2)} = \lim_{t\to 2^-} \frac{1}{5}\left[\ln\left|\frac{x-2}{x+3}\right|\right]_0^t$$

$$= \frac{1}{5}\lim_{t\to 2^-}\left(\ln\left|\frac{t-2}{t+3}\right| - \ln\frac{2}{3}\right) = \frac{1}{5}\left(\ln 0^+ - \ln\frac{2}{3}\right) = -\infty$$

$$\Rightarrow \int_0^2 \frac{dx}{x^2+x-6}\ (D) \Rightarrow \int_0^4 \frac{dx}{x^2+x-6}\ (D).$$

Q.6. $A = 2\pi\int x\,ds = 2\pi\int x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

$$= 2\pi\int_0^1 \sqrt{8y}\sqrt{1+\frac{2}{y}}\,dy = 4\pi\sqrt{2}\int_0^1\sqrt{y+2}\,dy$$

$$= 4\pi\sqrt{2}\,\frac{2}{3}(y+2)^{3/2}\Big|_0^1 = \frac{8\pi\sqrt{2}}{3}\left(\sqrt{27}-\sqrt{8}\right).$$

Q.7. $L = \int ds = \int_0^{2\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$ & $\dfrac{dr}{d\theta} = -2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\times\frac{1}{2}$

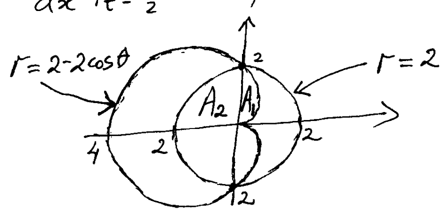
$$\Rightarrow L = \int_0^{2\pi}\sqrt{\cos^4\left(\tfrac{\theta}{2}\right)+\cos^2\left(\tfrac{\theta}{2}\right)\sin^2\left(\tfrac{\theta}{2}\right)}\,d\theta = \int_0^{2\pi}\left|\cos\frac{\theta}{2}\right|d\theta$$

$$\overset{t=\theta/2}{=} 2\int_0^{\pi}|\cos t|\,dt = 2\left[\int_0^{\pi/2}\cos t\,dt + \int_{\pi/2}^{\pi}(-\cos t)\,dt\right] = 4.$$

Q.8. $y - y_0 = m(x-x_0)$ where $x_0 = \frac{3\pi}{2}\cos\frac{3\pi}{2} = 0$, $y_0 = \frac{3\pi}{2}\sin\frac{3\pi}{2} = -\frac{3\pi}{2}$

and $m = \dfrac{dy}{dx}\Big|_{t=\frac{3\pi}{2}} = \dfrac{dy/dt}{dx/dt} = \dfrac{\sin t + t\cos t}{\cos t - t\sin t} = \dfrac{-1+0}{0+\frac{3\pi}{2}} = -\dfrac{2}{3\pi}.$

Q.9. a)



b) Area inside both curves $= 2(A_1+A_2)$,

where $A_1 = \frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}(2-2\cos\theta)^2\,d\theta$

and $A_2 = \frac{1}{2}\int_{\pi/2}^{\pi} r^2\,d\theta = \frac{1}{2}\int_{\pi/2}^{\pi} 4\,d\theta.$